

# Activity autocorrelation in financial markets

## A comparative study between several models

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**Abstract.** We study the activity of financial markets, i.e., the number of transactions per unit of time. Using the diffusion entropy technique we show that the autocorrelation of the activity is caused by the presence of peaks whose time distances are distributed following an asymptotic power-law which ultimately recovers an exponential behavior. We discuss these results in comparison with ARCH models, stochastic volatility models and multi-agent models showing that ARCH and stochastic volatility models better describe the observed experimental evidences.

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## 1 Overview

Financial time series present a strongly inhomogeneous time behavior. This is specially true when one considers either the volatility or the activity, the latter defined as the number of transactions per unit of time [1, 2]. Indeed if we look at the variance of the return in a time window of, say, one day we will observe periods of relative constant and regular behavior followed by other periods of strong variation of the price. In the same way there are days with few transactions and others where the number of trades is considerably larger. This great variability in the volatility or in the activity is generally referred to as volatility clustering or intermittency of volatility and activity. In this work we refer to both quantities. We will thus perform measures on the activity and use two volatility models: (i) ARCH models [3] and (ii) stochastic volatility (SV) models [4, 5], where the relationship between volatility and activity is set by the usual assumption of proportionality between them [6–10].

As is well known, the time interval or distance between two consecutive transactions  $\tau$  is a random variable described by a probability density function (pdf)  $\psi(\tau)$  which in many cases presents an asymptotic power law of the form [11–13]

$$\psi(\tau) \sim 1/\tau^\beta. \quad (1)$$

Note that the form of  $\psi(\tau)$  does not tell anything about the independence of consecutive  $\tau$ 's. However, if consecutive  $\tau$ 's are independent a power-law tail in  $\psi(\tau)$  can explain an inhomogeneous behavior in the number of events per unit of time, where a possible measure of this inhomogeneity is the distribution of the number of trades in a fixed period of time  $t$ . Thus, as Feller proved many years ago [14], if the time interval  $\tau$  between some particular events, which we will call markers, in a time series is distributed according to a given density  $\psi(\tau)$  and the independence condition  $\langle \tau_i \tau_j \rangle = \langle \tau_i \rangle \langle \tau_j \rangle$  for  $i \neq j$  holds, then the probability distribution to observe a fixed number of these markers  $y$  in a given time interval,  $p(y, t)$ , obeys the following scaling law

$$p(y, t) = \frac{1}{t^\delta} F\left(\frac{y}{t^\delta}\right), \quad (2)$$

where  $\delta$  is some positive exponent and  $F(x)$  is a positive and integrable function. With the help of a recently developed technique for the analysis of time series called Diffusion Entropy (DE) [15], we will see that the scaling observed in the distribution of the number of transactions in a time interval does not correspond to Feller's analytical prescription obtained with the density  $\psi(\tau)$  estimated from data.

We are thus forced to make the additional hypothesis that consecutive  $\tau$ 's are not independent. We will also assume that this correlation is due to the presence of peaks (or clusters) in the mean activity followed by periods of relative calm. Therefore the  $\tau_i$ 's are positively correlated because during a peak of activity they are shorter than the

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mean value while away from a peak they are greater than the mean. Indeed, in such a case  $\langle(\tau_i - \langle\tau_i\rangle)(\tau_j - \langle\tau_j\rangle)\rangle \geq 0$  which implies a positive correlation:  $\langle\tau_i\tau_j\rangle \geq \langle\tau_i\rangle\langle\tau_j\rangle$ .

Let  $\tau_c$  be the random time interval between two consecutive peaks and denote by  $\phi(\tau_c)$  its probability density function. Similarly to the distribution of the time interval between two consecutive transactions  $\psi(\tau)$  given by equation (1), we will also assume that  $\phi(\tau_c)$  obeys an asymptotic power law:

$$\phi(\tau_c) \sim 1/\tau_c^\mu. \quad (3)$$

In this scheme the results of the DE technique can be described directly in terms of the time interval and the magnitude of the peaks of activity, the latter described by a pdf  $h(x)$ . We will see, like in reference [16], that the distribution of the size of a cluster given by  $h(x)$  does not play an important role, because the time interval distribution  $\phi(\tau_c)$  is characterized by a more anomalous exponent than that of  $h(x)$ . Consequently we will interpret the results of DE as a consequence of a non-Poissonian distribution of the distance between peaks of activity.

There are in the literature several approaches which try to explain the autocorrelation of activity and volatility. One recent model [8,9,17] is based on the hypothesis that the intermittency of activity is caused by a subordination to a random walk, like in the case of the so called on-off intermittency [18]. As is clearly described in [9], this procedure should give for the distribution of time distances between clusters a scaling law of the form

$$\phi(\tau_c) \simeq \frac{1}{\tau_c^{3/2}} f\left(\frac{\tau_c}{\lambda}\right), \quad (4)$$

where  $f(t)$  is a cutoff function ensuring the existence of the first moment of  $\phi(\tau_c)$  and  $\lambda$  is the time scale at which this cutoff takes place. One simple choice for  $f(t)$  is given by the exponential  $f(t) = ke^{-t}$  which allows that  $\phi(\tau_c)$  presents an asymptotic Poissonian (i.e., exponential) behavior.

As we have mentioned, other possible approaches are provided by ARCH and SV models. We will show that both models lead to a correlation in the volatility which more likely resembles to some of the power-law tail exponents observed in a variety of financial markets. We will also show that a particular ARCH model, the TARARCH model presented in [19], and the SV model presented in [5,20] both result in the same scaling law than that observed with the DE technique. Finally, and due to the absence of intra-day disturbances in the time series obtained by ARCH and SV models, we are also able to evaluate numerically the waiting time distribution  $\phi(\tau_c)$  of the distance between peaks and this distribution is compatible with the empirical evidence given by a power-law behavior for a long transient period followed by a Poissonian behavior. We incidentally note that the asymptotic exponential behavior indicates that very far clusters do not influence each other.

The paper is organized as follows. We start with a brief review on the DE technique and a simple analytical proof that the DE results are determined by the most anomalous power-law tail between  $\phi(\tau_c)$  and  $h(x)$ . After that we

show the results obtained by means of the DE technique on tick by tick data of a Foreign Exchange (FX) market. We also perform a filtering procedure on data in order to prove that the observed scaling is due to the anomaly in the waiting time pdf  $\phi(\tau_c)$  rather than in the cluster size pdf  $h(x)$ . Finally, we briefly describe the ARCH and the SV models and the results of the DE and the waiting time distribution on the time series constructed using these models.

## 2 Diffusion entropy analysis

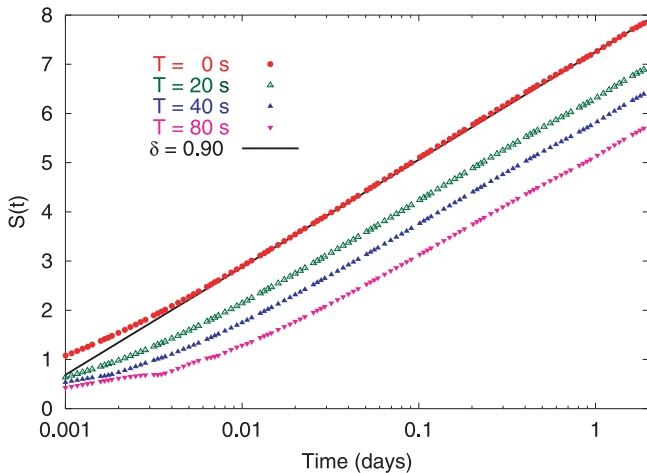
The diffusion entropy technique is basically an algorithm designed to detect memory in time series [15]. DE is specially suitable for intermittent signals, i.e., for time series where bursts of activity are separated by periods of quiescent and regular behavior. The technique has been designed to study the time distribution of some markers (or events) along the time series and thus discover whether these events satisfy the independence condition  $\langle\tau_i\tau_j\rangle = \langle\tau_i\rangle\langle\tau_j\rangle$  ( $i \neq j$ ) where  $\tau_i$  is the time interval between the marker labeled  $i - 1$  and the next one  $i$  [16,21–23]. As marker we use here a very simple definition: *each trade in the time series is a marker*. In order to apply the DE technique we need to construct a new series  $\xi_i$  which is a function of a coarse grained time  $i\Delta t$  (in our case  $\Delta t = 1$  s) and where  $\xi_i$  is precisely the number of transactions that occurred in the previous second. We next define a new random process through the following moving counting on  $\xi_i$

$$y_l(t) = \sum_{i=l-t/\Delta t}^l \xi_i. \quad (5)$$

Note that  $y_l(t)$  is precisely the number of markers (i.e., trades) in an interval of length  $t$  starting at position  $l$ . If we vary the value of  $l$  along the interval  $[0, N - t/\Delta t]$ , where  $N$  is the total length of the sequence, we can obtain the probability density function,  $p(y, t)$ , of this random process. It has been shown in [15] that for the zero-mean process  $y \rightarrow y - \langle y(t) \rangle$  and assuming that  $\xi_i$  is a renewal process  $p(y, t)$  obeys Feller's scaling law given by equation (2). Hence the entropy of this random process reads [15]

$$S(t) = - \int_{-\infty}^{\infty} p(y, t) \ln [p(y, t)] dy = A + \delta \ln(t). \quad (6)$$

From the slope of  $S(t)$ , in a logarithmic scale, we get an estimation of the scaling parameter  $\delta$ . In Figure 1 we show the results obtained with DE on tick by tick data of the US dollar – Deutsche mark futures market from 1993 to 1997 with a total of  $1.3 \times 10^6$  data points (solid circles). Other technical details regarding the data are further developed in [24]. The fit gives  $\delta = 0.90$ . As is shown in [15] if condition  $\langle\tau_i\tau_j\rangle = \langle\tau_i\rangle\langle\tau_j\rangle$  holds then there exists a relation between the scaling exponent  $\delta$  and the exponent  $\beta$



**Fig. 1.** Results of the DE analysis for US dollar – Deutsche mark futures market for different values of the time-threshold  $T$ .

of the power-law tail of the time distribution of markers (in our case, trades)  $\psi(\tau)$  (see Eq. (1)). This relation reads

$$\delta = \begin{cases} 1/(\beta - 1), & \text{if } 2 < \beta < 3; \\ 0.5, & \text{if } \beta > 3. \end{cases} \quad (7)$$

We have shown elsewhere that for the FX market under consideration the power-law tail exponent of the waiting time distribution between trades is near 3.5 [13]. According to equation (7) this would lead to  $\delta = 0.5$  — after a transient comparable with the mean trade distance of  $\langle \tau \rangle = 23.6$  s — in disagreement with the value  $\delta = 0.90$  obtained by DE.

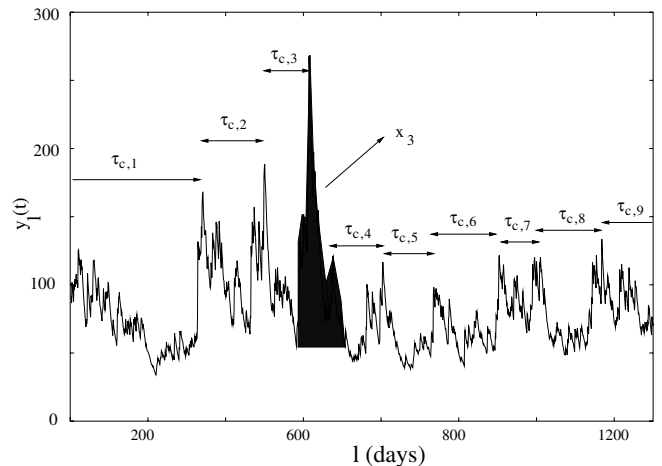
We now present a picture which describes peaks of activity separated by periods with a low number of transactions. We see in Figure 2 a schematic representation of this picture in which, for instance, the intensity  $x_3$  of the third peak is represented by a black spot and this corresponds to the total number of transactions attributed to this peak.

We suppose that the time intervals between peaks,  $\tau_c$ , are distributed according to a pdf  $\phi(\tau_c)$  which asymptotically behaves as in equation (3), i.e.,  $\phi(\tau_c) \sim 1/\tau_c^\mu$ . We also assume that the intensity of a given peak,  $x_i$ , defined as the total number of transactions in the peak, has a distribution given by a pdf  $h(x)$ . In the context of earthquakes  $h(x)$  is generally referred to as the Pareto law of the size of the earthquake clusters, since asymptotically  $h(x) \sim 1/x^{\alpha+1}$  as in the Pareto distribution [16].

We will now present a proof that the DE technique perceives the most anomalous (i.e., the smallest) of the two exponents  $\mu$  and  $\alpha + 1$ . Indeed, let  $\rho(x, t)$  be the joint pdf for the waiting time  $\tau_c$  between clusters and their intensity  $x$ . We denote by

$$\hat{\rho}(\omega, s) = \int_{-\infty}^{\infty} dx e^{i\omega x} \int_0^{\infty} d\tau_c e^{-s\tau_c} \rho(x, \tau_c)$$

its Fourier-Laplace transform. Observe that in terms of  $\hat{\rho}(\omega, s)$  the Laplace transform of the waiting time distribu-



**Fig. 2.** Scheme of the model for the anomalous scaling for activity and volatility.

tion  $\hat{\phi}(s)$  and the Fourier transform of the size distribution  $\tilde{h}(\omega)$  are given by

$$\hat{\phi}(s) = \hat{\rho}(\omega = 0, s) \quad \text{and} \quad \tilde{h}(\omega) = \hat{\rho}(\omega, s = 0). \quad (8)$$

We assume that the time duration of a peak is negligible with respect to the mean time interval between peaks [25]. In such a case we can use the Continuous Time Random Walk (CTRW) formalism to calculate the probability density function,  $p(y, t)$ , that the number of trades at time  $t$  is given by  $y$ . Thus, in terms of the joint distribution  $\hat{\rho}(\omega, s)$  the Fourier-Laplace transform of  $p(y, t)$  is given by [13]

$$\hat{p}(\omega, s) = \frac{1 - \hat{\phi}(s)}{s} \frac{1}{1 - \hat{\rho}(\omega, s)}. \quad (9)$$

We can easily see that as  $s \rightarrow 0$  then  $[1 - \hat{\phi}(s)]/s \sim \langle \tau_c \rangle$ , where  $\langle \tau_c \rangle$  is the mean waiting time. Hence

$$\hat{p}(\omega, s) \simeq \frac{\langle \tau_c \rangle}{1 - \hat{\rho}(\omega, s)}, \quad (s \rightarrow 0). \quad (10)$$

On the other hand, as  $s \rightarrow 0$  we have (see Eq. (3))

$$\hat{\phi}(s) \simeq 1 - \langle \tau_c \rangle s + c_0 s^{\mu-1}, \quad (2 < \mu < 3). \quad (11)$$

Moreover, as  $\omega \rightarrow 0$  and consistently with the Pareto law according to which  $h(x)$  decays as  $1/x^{\alpha+1}$ , we have

$$\tilde{h}(\omega) \simeq 1 + i\langle x \rangle \omega + b_0 \omega^\alpha, \quad (1 < \alpha < 2), \quad (12)$$

where  $\langle x \rangle$  is the average peak intensity. Taking into account equations (11, 12) we see that as  $s \rightarrow 0$  and  $\omega \rightarrow 0$  the joint distribution  $\hat{\rho}(\omega, s)$  can be written as

$$\hat{\rho}(\omega, s) \simeq 1 - \langle \tau_c \rangle s + i\langle x \rangle \omega - i\langle x \tau_c \rangle \omega s + b(s) \omega^\alpha + c(\omega) s^{\mu-1}, \quad (13)$$

where  $b(s)$  and  $c(\omega)$  are such that  $b(0) = b_0$  and  $c(0) = c_0$ . We recall that the DE technique measures the scaling in a “moving reference frame” where the average activity is zero,  $\langle y(t) \rangle = 0$  for all  $t \geq 0$ . In order to obtain the

pdf for  $y(t)$  in such reference frame we perform in equation (13) the following substitution

$$s \longrightarrow s + i\omega \frac{\langle x \rangle}{\langle \tau_c \rangle}, \quad (14)$$

and after applying the diffusive limit  $\langle \tau_c \rangle |s| \ll \langle x \rangle |\omega|$  we get, to the lowest order,

$$\hat{p}(\omega, s) \simeq 1 - \langle \tau_c \rangle s + b_0 \omega^\alpha + c_0 (i\langle x \rangle / \langle \tau_c \rangle)^{\mu-1} \omega^{\mu-1}. \quad (15)$$

Substituting equation (15) into equation (10) finally yields

$$\hat{p}(\omega, s) \simeq \frac{\langle \tau_c \rangle}{s \langle \tau_c \rangle - b\omega^\alpha - c(i\langle x \rangle / \langle \tau_c \rangle)^{\mu-1} \omega^{\mu-1}}. \quad (16)$$

This equation shows that the smallest exponent between  $\alpha$  and  $\mu - 1$  determines the asymptotic scaling of  $p(y, t)$  according to the exponent

$$\delta = \begin{cases} 1/(\mu - 1), & \text{if } \mu - 1 > \alpha; \\ 1/\alpha, & \text{if } \mu - 1 < \alpha. \end{cases} \quad (17)$$

Therefore, the scaling perceived by DE is determined by the most anomalous exponent of the scaling between the size of the clusters of activity and the distribution of their time intervals. Note that the case  $\delta = 1/(\mu - 1)$  agrees with that of equation (7). We also observe that we have proven this fundamental result for the most general case in which there is no assumption on the possible correlation, or independence, among intensities and waiting times.

Having this in mind, we return to the problem of explaining the scaling exponent  $\delta = 0.90$  appearing in the US dollar – Deutsche mark futures market. To what effect is due this scaling? In other words, is the exponent  $\delta$  determined by the time interval between clusters or by their size? In order to solve this question we impose a cutoff in the size of the peaks of activity by eliminating those transactions whose time interval from the previous one is below certain threshold  $T$  (note that this actually reduces cluster sizes because the number of transactions counted is now smaller). If after this cutoff procedure the scaling remains invariant then  $\delta$  would be determined by the time intervals and not by the size of the clusters. In Figure 1 the DE results are shown for different values of the time-threshold  $T$  ranging from 0 to 80 s. We see there that the slope is practically unchanged which confirms the assumption that the exponent  $\delta = 0.90$  is solely determined by the anomaly in the time intervals between the clusters and not by any anomaly of their size.

### 3 ARCH, stochastic volatility and on-off intermittency models

At the end of the last section, we have indirectly shown that the anomalous scaling  $\delta = 0.90$  observed in data is not caused by fat tails in the peak intensity distribution  $h(x)$  but by the anomalous scaling in the waiting time

distribution between peaks. Another more direct way to prove this would have been to single out the peaks on actual data and look for their time distribution. Unfortunately it is very difficult, on real data, to define a peak of activity and compute the waiting time distribution between them. This is because there are peaks of activity that appear at fixed times (we will call these “deterministic peaks”) at the daily opening and closing sessions, at the opening during the day of other markets and even weekly at the opening of each monday [26]. These deterministic peaks do not contribute to the increase of entropy. However, they do affect any estimation of the waiting time distribution which makes it very difficult obtaining a reliable estimate of it.

A possible way out from this situation would be to generate an artificial time series simulating the real market evolution. In this artificial series, the activity would be replaced by the volatility following the accepted correspondence between them [6–10] and we would check there all the scaling phenomena reported up till now. Specifically we will assume the proportionality between activity and volatility (see Eq. (19) below). In fact, in order to explain the observed scaling, it is not necessary to assume the exact proportionality. Indeed, the crucial point is that the occurrence of volatility peaks correspond to peaks for the activity, being the scaling determined by the statistics of the time intervals between peaks.

We will follow this procedure and choose two well accepted models for reconstructing market activity without deterministic peaks: (i) the TARCh model [19], and (ii) the stochastic volatility model presented in [20]. We will see that both models give the same results than those of the DE technique. We finally discuss the prescription given in equation (4) based on multi-agent models to see whether it agrees with our results or not.

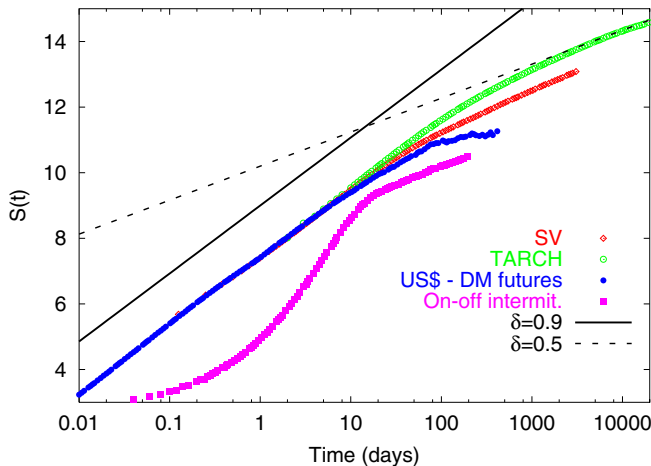
#### 3.1 The TARCh model

This particular ARCH model, called TARCh by its authors [19], is given by

$$\begin{aligned} \sigma_t^2 &= k + \alpha R_{t-1}^2 + \beta \sigma_{t-1}^2 + \gamma \Theta(-R_{t-1}) R_{t-1}^2 \\ R_t &= \sigma_t \eta_t \end{aligned} \quad (18)$$

where  $\sigma_t$  is the volatility,  $R_t$  is the one day return calculated at time  $t$  (which implies that the time step of the model is  $t_0 = 1$  day),  $\Theta(x)$  is the Heavyside step function and  $\eta_t$  is Gaussian noise with zero mean and unit variance. The other parameters, estimated from daily data of the Dow Jones Industrial Average Index from 1988 to 2000 and reported in [19], are:  $k = 0.0184$ ,  $\alpha = 0.0151$ ,  $\gamma = 0.0654$ ,  $\beta = 0.9282$  [27]. Using equation (18) we generate a time series for  $\sigma_t$ . We then perform the DE analysis on this series by defining, with the notation of equation (5), the process

$$y_l(nt_0) = c \sum_{i=l-nt_0}^l \sigma_i, \quad (19)$$

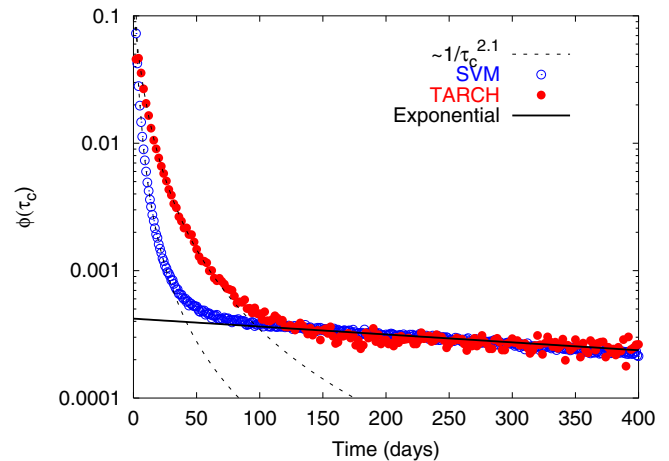


**Fig. 3.** Results of the DE analysis for US dollar – Deutsche mark futures with  $T = 0$  (solid circles), the TARCH model (empty circles), the SV model (diamonds), and finally for the on-off intermittency model as given by equation (4) (solid squares).

where  $t_0$  is the time step of the model,  $n$  is a positive integer,  $l$  is an integer multiple of  $t_0$ ,  $\sigma_i$  is the volatility and  $c$  is a proportionality constant. The results are shown in Figure 3 (empty circles) compared with the results on real data for  $T = 0$  (solid circles). We clearly see that the TARCH model predicts for  $t < t_P$  a scaling exponent  $\delta = 0.90$  which agrees with actual data. For  $t$  greater than a Poissonian time  $t_P \approx 100$  days the model yields  $\delta = 0.5$ . It is worth noticing that the change in the slope of real data is likely due to the lack of statistics. Moreover, we do not have enough data points to determine whether the change of slope takes place at the same time scale than in the TARCH model. Nevertheless, ARCH-type models (similar results were obtained with  $\gamma = 0$  in Eq. (18)) seem to take into account the correct structure of the intermittency of financial series.

We should now stress the fact that the value of the constant  $c$  which appears in equation (19) is somewhat arbitrary depending on the details of the model and on the value of  $t_0$ . Nevertheless we perform different numerical test showing that changing the value of  $c$  modifies the absolute value of the diffusion entropy but it does not change the slope  $\delta$  (in some sense changing  $c$  means a different choice for the binning used to calculate the diffusion entropy). In Figure 3 we choose the value of  $c$  in order to superimpose the graph obtained from the TARCH model on the one coming from the real data. The same is done in the next section for the SV model. We also observe that the results of the DE analysis remain unchanged if in equation (19) we replace  $\sigma_i$  by  $\sigma_i^2$ .

From the series generated using equation (18) we can also evaluate the waiting time distribution between peaks because now we do not have deterministic peaks and other periodic effects that are present in real data. The result is shown in Figure 4 and as we see there that for  $\tau_c < t_P$  a



**Fig. 4.** Waiting time distribution for the distance between clusters  $\tau_c$  (in logarithmic scale) for the TARCH model (solid circles) and the SV model (empty circles). A distinct asymptotic behavior is clearly present: a power-law tail with exponent  $\mu \simeq 2.1$  and an exponential decay.

good fit is provided by the following power-law:

$$\phi(\tau_c) \simeq \frac{(\mu - 1)\nu}{(1 + \nu\tau_c)^\mu}, \quad (20)$$

where  $\nu = 0.1 \text{ day}^{-1}$  and  $\mu = 2.1$ . For  $\tau_c > t_P$  a clear exponential (Poisson) behavior is present. This result has a simple physical explanation: if there is a first cluster at time  $t$  the probability to observe another one just after the first is high while very distant clusters are practically independent which explains the asymptotic Poissonian behavior that we observe in Figure 4.

### 3.2 An stochastic volatility model

There exists another way of modelling volatility clustering. The so-called stochastic volatility models [4, 5, 20] are an alternative choice to ARCH models and they are considered to be the most natural extension to the classic geometric Brownian motion for the price dynamics in continuous time finance. Let us start with the zero-mean return  $X(t)$  (i.e., the detrended log-price) the dynamics of which is governed by the following stochastic differential equation

$$\dot{X}(t) = \sigma\xi_1(t), \quad (21)$$

where this equation has to be understood in the Itô sense,  $\sigma$  is the volatility and  $\xi_1 dt$  is the Wiener process, i.e.,  $\xi_1(t)$  is Gaussian white noise with zero mean and correlation function given by

$$\langle \xi_1(t)\xi_1(t') \rangle = \delta(t - t'). \quad (22)$$

All SV models assume that the volatility  $\sigma$  is itself a random process. There are several ways to describe the dynamics of the volatility [4]. One of the simplest models, which still contains almost all the basic ingredients

prescribed by real markets, is given by the Ornstein-Uhlenbeck (OU) process [20]

$$\dot{\sigma}(t) = -a[\sigma(t) - m] + k\xi_2(t). \quad (23)$$

One key property of this model is that it exhibits a stationary solution due to the reverting force — quantified by  $a$  — to a certain average  $m$ , the so-called “normal level of volatility”. The stationary solution is a Gaussian distribution and the resulting distribution for the return has fat tails [20]. In addition, stylized facts such as the negative skewness and the leverage correlation [5,20] require that the changes of the volatility be negatively correlated with the random source of return changes. In other words, the driving noises appearing in equations (21, 23) are anti correlated, that is:

$$\langle \xi_1(t)\xi_2(t') \rangle = \rho\delta(t - t')$$

where  $-1 \leq \rho \leq 0$ . For the OU SV model as given in equation (23) the characteristic exponential time decay of the leverage correlation is given by  $1/a$  which is typically of the order of few trading days (see below).

Although the OU model has some disagreements with some experimental observations [28], it is complex enough to catch all the statistical properties that we are studying here. We therefore simulate the SV model with the parameters estimated from daily data of the Dow Jones Index from 1900 to 1999. Thus, the reverting force is equal to  $a = 0.05 \text{ days}^{-1}$ , the noise amplitude is  $k = 0.0014 \text{ days}^{-1}$ , the normal level of volatility reads  $m = 0.011 \text{ days}^{-1/2}$  and the correlation coefficient is  $\rho = -0.5$  [20]. We set the time step  $t_0 = 1$  hour. The results of the DE analysis are reported in Figure 3 (diamonds) while the waiting time distribution between clusters is shown in Figure 4 (empty circles). In this case, and analogously to the TARARCH model, we also observe a power-law behavior followed by an exponential decay. The only difference lies in the value of the Poissonian time  $t_P$  which for this model is near to 40 days while for the TARARCH model is approximately 100 days. We have checked numerically that this difference is due to the fact that the parameters defining each model are estimated from the Dow Jones index *over different periods of time*, much larger for the SV model than for the TARARCH model. In any case, we cannot discard any of these approaches on the basis of the empirical results.

### 3.3 On-off intermittency models

The intermittent model of the activity is also predicted and studied by several multi-agent or minority game models [8,9,29]. These models can be connected to on-off intermittency and they generally imply that the persistence of activity is subordinated to a random walk which indicates that the waiting time distribution has the form given in equation (4). As suggested in [8,9] we also obtain the so-called variogram of the data, although the results denote that, if a  $3/2$  tail is present like in equation (4), causing a  $\sqrt{t}$  behavior in the variogram, the tail lasts less than 5 days probably because the FX market is more liquid than the

ones considered in [8,9]. Furthermore the DE analysis performed on a series generated according to equation (4) also leads to a transient followed by the exponent  $\delta = 0.5$  of the asymptotic behavior whereas, as is clearly seen in Figure 3, the transient never exhibits the exponent  $\delta = 0.90$  beyond 1 day but presents a constantly increasing exponent which is most of the time greater than 1.

## 4 Conclusions

We have performed the DE analysis on the activity of the tick by tick time series US dollar – Deutsche mark futures from 1993 to 1997. The results clearly show the presence of an anomalous scaling, for the probability distribution of the activity,  $p(y, t)$ , near the exponent  $\delta = 0.90$ . We have also implemented the same analysis on the volatility obtained with the TARARCH model and with the OU stochastic volatility model. We find in both cases an excellent agreement between the scaling measured either on actual data and on the constructed series.

We compare the results with the scheme of the subordination of the volatility to a random walk leading to equation (4) observing that a power-law exponent  $\mu \approx 2$  for the tail of the distribution of the distances between peaks  $\phi(\tau_c)$  of volatility is more plausible. We believe that the main reason why the TARARCH and the SV models give better results is that in on-off intermittency models the occurrence of a peak can be considered as subordinated to a random walk but the weak restoring force (which has to be included in the model in order to describe mean reversion) not only causes the final stationarity and the exponential tail of Figure 4 but also affects the process of regression to equilibrium modifying it in a fundamental way (from  $\mu = 3/2$  to  $\mu \simeq 2$ ) the transient behavior of  $\phi(\tau_c)$ .

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## References

1. In other works the activity is defined as the number of transactions per unit of time weighted with the corresponding volume (see for instance Ref. [2]). In this paper we focus on the properties of the transaction frequency, i.e., the number of transactions per unit time regardless the volume. In any case and for simplicity we will use the terms activity and transaction frequency without distinction
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24. The data was provided by the *The Institute for Financial Markets* and consist of transaction prices. We use the data of the closest-to-maturity contract. Since we do not know the exact expiration date, i.e., the last traded day for each contract, we have discarded some usable data, in the period comprised between the expiration day and the end of the month. We also analyse a different prescription consisting on switching between contracts at the very beginning of the maturity month, obtaining identical results. We use the most usual roll-over procedure consisting in a backward correction of the previous prices. This technique is intended to correct the gap that usually exists between the last traded price of the old contract, and the first traded price of the new one the following morning. We do not perform any outlier correction. Regarding overnight dynamics we do not know whether nights and days follow the same dynamics, or whether ordinary nights and weekends are equivalents. Thus we use the simplest approach and neglect those periods of time for which we have no data
25. We have checked numerically that this hypothesis should not be considered as a real constraint. Even in the case that the typical time duration of a peak is comparable with the distance between peaks, the condition required in DE technique for obtaining the correct asymptotic scaling is that  $t \gg \langle \tau_c \rangle$
26. This periodic behavior can be easily singled out by a fast Fourier transform of the FX data
27. In the original model there is a mean drift in the return corresponding to the mean growth of the market. Our data correspond to a FX market where this effect is not present. Therefore, there is no mean drift in our analysis
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